SECONDARY FLOWS AND FLUID INSTABILITY BETWEEN ROTATING CYLINDERS

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Experiments indicate that when Couette flow between rotating cylinders becomes unstable, a new steady flow arises. Mathematically this means that the corresponding steady-state boundary value problem for the Navier-Stokes equations has more than one solution. The purpose of the present study is to prove this fact in the case where the cylinders are rotating in the same direction. The indicated phenomenon occurs not only for Couette flow, but also for a certain class of fluid flows.

The method employed here is based on Krasnosel'skii's theorem [1] on the bifurcation points of operator equations. The application of this theorem to Navier-Stokes equations was considered in [2], where the nonuniqueness of the solution of a certain steady-state spatially periodic problem was demonstrated.

The most difficult task involved in the application of Krasnosel'skii's theorem is the investigation of the spectra of linearized problems. In the case we are about to consider the study of the spectrum is facilitated by the results of Krein and Gantmakher on oscillatory integral operators [3 to 5].

The principal conclusions concerning bifurcation are formulated in Theorem 4.1 and in the notes made in connection with it.

We shall also show that the flows under consideration are unstable for large Reynolds numbers (see Theorem 5.1).

1. Formulation of the problem. Let any viscous incompressible homogeneous fluid fill the cavity between two coaxial cylinders with $r = r_1$ and $r = r_2$; r, θ, z are cylindrical coordinates. We shall attempt to find the axisymmetrical steady-state flows, i.e. flows such that the velocity components v_r', v_θ', v_z' depend solely on r and z and are independent of θ . We shall also assume that v_r', v_θ', v_z' are periodic relative to z with a period $2\pi/\alpha_0$, and that the velocity flux through the transverse cross section of the cavity is 0

$$\int_{r_1}^{r_2} v_z'(r, z)\, r\, dr = 0 \qquad (1.1)$$

Assuming that the cylinders are solid and rotate with the angular velocities ω_1 and ω_2, respectively, and that the vector of the vortical mass forces F is of the form $(0, \nu F(r), 0)$, it is easy to see that all of the requirements posed are satisfied by flow with a velocity vector $\mathbf{v}_0$ and a pressure P_0

$$\begin{pmatrix} v_{0r} = v_{0z} = 0 \\ v_{0\theta} = v_0(r) \end{pmatrix}, \qquad P_0 = \int_{r_1}^{r} \frac{v_0^2(\rho)}{\rho}\, d\rho + \text{const} \tag{1.2}$$

where the function $v_0(r)$ is defined unambiguously as the solution of the boundary value problem

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) v_0 = -F(r) \qquad \begin{pmatrix} v_0(r_1) = \omega_1 r_1 \\ v_0(r_2) = \omega_2 r_2 \end{pmatrix} \tag{1.3}$$

We further assume that F, and therefore v_0, do not depend on the coefficient of viscosity ν. Specifically, if $F = 0$, then (1.2) represents Couette flow,

$$v_0(r) = ar + \frac{b}{r}, \qquad a = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}, \qquad b = \frac{(\omega_1 - \omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2} \tag{1.4}$$

Seeking solutions v', P' of the formulated problem which differ from (1.3) in the form

$$\mathbf{v}' = \mathbf{v} + \mathbf{v}_0, \qquad P' = \nu p + P_0 \tag{1.5}$$

we obtain the following system of equations for determining $\mathbf{v}$, p:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 \tag{1.6}$$

$$\Delta v_r - \frac{v_r}{r^2} - \frac{\partial p}{\partial r} = \lambda\left[v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - 1\frac{v_\theta^2}{r} - 2\frac{v_0}{r} v_\theta\right]$$

$$\Delta v_\theta - \frac{v_\theta}{r^2} = \lambda\left[v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} + \left(\frac{dv_\theta}{dr} + \frac{v_0}{r}\right) v_r\right]$$

$$\Delta v_z - \frac{\partial p}{\partial z} = \lambda\left[v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\right] \qquad \left(\lambda = \frac{1}{\nu},\ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

Here the functions v_r, v_θ, v_z must be $2\pi/\alpha_0$-periodic relative to z and vanish for $r = r_1, r_2$. The fulfillment of the condition

$$\int_{r_1}^{r_2} v_z(r, z) r\, dr = 0 \tag{1.7}$$

which follows from (1.1), is also required.

The linearized problem which corresponds to problem (1.6) to (1.7) is of the form

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0, \qquad \Delta u_r - \frac{u_r}{r^2} - \frac{\partial q}{\partial r} = -\lambda \frac{2 v_0}{r} u_\theta \tag{1.8}$$

$$\Delta u_\theta - \frac{u_\theta}{r^2} = \lambda\left(\frac{dv_0}{\partial r} + \frac{v_0}{r}\right) u_r, \qquad \Delta u_z - \frac{\partial q}{\partial z} = 0, \qquad \int_{r_1}^{r_2} u_z r\, dr = 0$$

and the conjugate problem is

$$\frac{1}{r}\frac{\partial}{\partial r}(rw_r)+\frac{\partial w_z}{\partial z}=0, \qquad \Delta w_r-\frac{w_r}{r^2}-\frac{\partial Q}{\partial r}=\lambda\left(\frac{dv_0}{dr}+\frac{v_0}{r}\right)w_\theta \quad (1.9)$$

$$\Delta w_\theta-\frac{w_\theta}{r^2}=-\lambda 2\frac{v_0}{r}w_r, \qquad \Delta w_z-\frac{\partial Q}{\partial z}=0, \qquad \int_{r_1}^{r_2} w_z r\,dr=0$$

The boundary conditions for the vectors $\mathbf{u}$, $\mathbf{w}$ are the same as for the vector $\mathbf{v}'$.

Let us consider the set N of doubly continuously differentiated solenoidal vectors $\{\mathbf{v}\}$ which are defined in the closed domain $\{r_1 \le r \le r_2\ ;\ -\infty < z < +\infty\}$, are axisymmetrical ($v_r$, v_θ, v_z are independent of θ), vanish for $r = r_1$, r_2 have a flux of zero through the transverse cross section of the cavity, and are such that v_r, v_θ are even functions of z, and v_z is odd. By $H_1{}^\circ$ we denote the Hilbert space obtained by supplementing the set N with respect to the norm generated by the scalar product

$$(\mathbf{v},\mathbf{u})_{H_1^\circ}=-\int_{-\pi/\alpha_0}^{\pi/\alpha_0}dz\int_{r_1}^{r}\Delta\mathbf{v}\mathbf{u}r\,dr=$$

$$=-\int_{-\pi/\alpha_0}^{\pi/\alpha_0}dz\int_{r_1}^{r_2}\left[\left(\Delta v_r-\frac{v_r}{r^2}\right)u_r+\left(\Delta v_\theta-\frac{v_\theta}{r^2}\right)u_\theta+\Delta v_z u_z\right]r\,dr.$$

Inverting the operator defined by relations (1.6) and (1.7) for $\lambda = 0$, we reduce problem (1.6),(1.7) to the operator equation

$$\mathbf{v}=\lambda K_0\mathbf{v} \tag{1.10}$$

In a similar way, problems (1.8) and (1.9) are reducible to the operator equations

$$\mathbf{u}=\lambda A_0\mathbf{u}, \qquad \mathbf{w}=\lambda A_0{}^*\mathbf{w} \tag{1.11}$$

The operators K_0, A_0, $A_0{}^*$ are completely continuous in the space $H_1{}^\circ$; the operator A_0 is the Frechet differential of the operator K_0 at the point $\mathbf{v} = 0$; $A_0{}^*$ is the adjoint of A_0 in $H_1{}^\circ$. All of this follows from the results of [2]. We note that $H_1{}^\circ$ is a subspace of the space H_1 considered in [2], and that the operators K_0, A_0, $A_0{}^*$ are the contractions of the operators K, A, A^* of [2] onto the (invariant) subspace $H_1{}^\circ$.

2. Reduction to an integral equation. By expanding in a Fourier series we see that the solution of spectral problem (1.8) is a linear combination of solutions of the form

$$u_r=u(r)\cos\alpha z, \quad u_\theta=v(r)\cos\alpha z \qquad\qquad (2.1)$$
$$u_z=w(r)\sin\alpha z, \quad q=\varkappa(r)\cos\alpha z$$

where $\alpha = k\alpha_0$ (k is a natural number) and the functions w, $\varkappa$ are expressed in terms of u and v by Formulas

$$w(r)=-\frac{1}{\alpha r}\frac{d}{dr}(ru), \qquad \varkappa(r)=-\frac{1}{\alpha}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\alpha^2\right)w$$

The functions u, v and the corresponding eigenvalue of λ are determined by solving the spectral problem

$$(L-\alpha^2)^2 u = 2\alpha^2\lambda\omega v,\ (L-\alpha^2) v = -\lambda g u, \quad u = v = u' = 0 \quad (\text{for } r = r_1, r_2)$$

$$\omega(r) = \frac{v_0}{r}, \quad g(r) = -\left(\frac{dv_0}{dr} + \frac{v_0}{r}\right), \quad L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \quad (2.2)$$

Naturally enough, the aforementioned linear combination contains only those solutions of (2.1) which have the same corresponding value of λ. The functions ω and g will henceforth be considered continuous.

Now let us reduce problem (2.2) to an integral equation. Let $G_1(r, \rho)$, $G_2(r, \rho)$ be the Green's functions of the differential operators $-r(L-\alpha^2)$, $r(L-\alpha^2)^2$ with the boundary conditions $u = 0$ and $u = u' = 0$ $(r = r_1, r_2)$, respectively. Let G_1, G_2 be integral operators given by Formulas

$$G_k f = \int_{r_1}^{r_2} G_k(r, \rho) f(\rho) \rho \, d\rho \quad (2.3)$$

Both Green's functions $G_1(r, \rho)$ and $G_2(r, \rho)$ are continuous with respect to r, ρ and symmetrical. As we know, this follows from the symmetry of the corresponding differential operators.

Let us denote by H_0 the Hilbert space L_2 with the weight r on the segment $[r_1, r_2]$. The scalar product in H_0 is given by Formula

$$(\varphi, \psi)_{H_0} = \int_{r_1}^{r_2} \varphi(r)\psi(r) r dr \quad (2.4)$$

The operators G_1, G_2 defined by Formula (2.3) are symmetrical and completely continuous in H_0.

Problem (2.2) is equivalent to finding the spectrum of the system of integral equations

$$u = 2\alpha^2\lambda G_2\omega v, \qquad v = \lambda G_1 g u \quad (2.5)$$

or any of the integral equations

$$u = \mu G_2\omega G_1 g u, \qquad v = \mu G_1 g G_2 \omega v \qquad (\mu = 2\alpha^2\lambda^2) \quad (2.6)$$

Here ω, g denote the operators of multiplication by the functions $\omega(r)$ and $g(r)$, respectively.

Similarly, seeking the solution of conjugate problem (1.9) in the form

$$w_r = u_1(r)\cos\alpha z, \quad w_\theta = v_1(r)\cos\alpha z, \quad w_z = w_1(r)\sin\alpha z, \quad Q = \varkappa_1(r)\cos\alpha z$$

$$\left(w_1(r) = -\frac{1}{\alpha r}\frac{d}{dr}(r u_1), \qquad \varkappa_1(r) = -\frac{1}{\alpha}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \alpha^2\right) w_1\right) \quad (2.7)$$

we arrive at the problem of eigenvalues for determining u_1, v_1

$$(L-\alpha^2)^2 u_1 = \lambda\alpha^2 g v_1, \quad (L-\alpha^2) v_1 = -2\lambda\omega u_1$$
$$u_1 = u_1' = v_1 = 0 \quad (\text{for } r = r_1, r_2) \quad (2.8)$$

Problem 2.8 is equivalent to the system of integral equations

$$u_1 = \lambda\alpha^2 G_2 g v_1, \qquad v_1 = 2\lambda G_1 \omega u_1 \quad (2.9)$$

or to any of the integral equations

$$u_1 = \mu G_2 g G_1 \omega u_1, \qquad v_1 = \mu G_1 \omega G_2 g v_1 \tag{2.10}$$

For specificity we shall next consider the first equation of (2.6), which can be written as

$$u = \mu B u \qquad (B = G_2 \omega G_1 g) \tag{2.11}$$

Let $\mu > 0$ be one of its eigenvalues. Then $\lambda = \pm \sqrt{\mu/2\alpha^2}$ is the eigenvalue of problem (2.2); v can be found from the second formula of (2.5). The equation conjugate to (2.11) is of the form

$$u_0 = \mu B^* u_0 \qquad (B^* = g G_1 \omega G_2) \tag{2.12}$$

Combining the second equations of (2.10) and (2.12) we note that $u_0 = gv_1$ is the solution of Equation (2.12).

Let us suppose that $\mu > 0$ is a single eigenvalue of the operator B. Hence it follows (see [2], Lemma 1.5) that $(u, u_0)_{H_0} = (u, gv_1)_{H_0} \neq 0$.

Lemma 1.1. Let $\mu > 0$ be an eigenvalue of the operator $B = G_2 G_1 g$, and let its rank be 1. Then $\lambda = \mp \sqrt{\mu/2\alpha^2}$ is an eigenvalue of the operator A_0 (see (1.22)), and its rank is also 1.

Proof. Let us compute the scalar producy $(\mathbf{u}, \mathbf{w})_{H_1^\circ}$, where $\mathbf{u}, \mathbf{w}$ are the eigenvectors of the operators A_0, A_0^* defined by Equations (2.1) and (2.7) which correspond to the eigenvalue λ. Multiplying the second third and fourth equations of (1.8) by w_r, w_θ and w_z, respectively, we find that

$$(\mathbf{u}, \mathbf{w})_{H_1^\circ} = \lambda \int_{-\pi/\alpha_0}^{\pi/\alpha_0} \int_{r_1}^{r_2} (2\omega u_\theta w_r + g u_r w_\theta)\, r dr dz \tag{2.13}$$

With the aid of (2.1) and (2.7) the second equation and the first equation of (2.9), we find from (2.13) that

$$(\bar{\mathbf{u}}, \bar{\mathbf{w}})_{H_1^\circ} = \frac{\pi\lambda}{\alpha_0} \int_{r_1}^{r_2} (2\omega v u_1 + g u v_1)\, r dr = \frac{\pi\lambda}{\alpha_0} [(\mu\omega G_1 g u,\ G_2 g v_1)_{H_0} + (u,\ u_0)_{H_0}] =$$
$$= \frac{\pi\lambda}{\alpha_0} [(\mu G_2 \omega G_1 g u,\ g v_1)_{H_0} + (u, u_0)_{H_0}] = \frac{2\pi\lambda}{\alpha_0} (u, u_0)_{H_0} \neq 0$$

Hence it follows that the rank of the eigenvalue λ is unity. Lemma has been proved.

3. On Green's functions G_1 and G_2. Let us consider the differential operators $-r(L - \alpha^2)$, $r(L - \alpha^2)^2$, which can be represented in the form

$$\begin{aligned} -r(L - \alpha^2) u &= \rho_0 \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 u \\ r(L - \alpha^2)^2 u &= \rho_0 \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 \rho_0 \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 u \end{aligned} \qquad \begin{pmatrix} \rho_0(r) = \rho_2(r) = I_1(\alpha r) \\ \rho_1(r) = r / \rho_2^2 \end{pmatrix} \tag{3.1}$$

Here $\rho_0(r)$, $\rho_1(r)$, $\rho_2(r)$ are positive functions and I_1 is a modified Bessel function.

In accordance with results of Krein [3] this implies that $G_1(r, \rho)$, $G_2(r, \rho)$ are oscillatory kernels. This means that the following conditions are fulfilled:

1. $G_k(r, \rho) > 0 \qquad (r_1 < r, \rho < r_2)$

2. $\det \| G_k(\eta_i, \rho_s) \|_{i,s=1}^{n} \geq 0$ for $r_1 < \begin{matrix} \eta_1 < \eta_2 < \ldots < \eta_n \\ \rho_1 < \rho_2 < \ldots < \rho_n \end{matrix} < r_2$

3. $\det \| G_k(\rho_i, \rho_s) \|_{i,s=1}^{n} > 0$ for $r_1 < \rho_1 < \ldots < \rho_n < r_2$

An integral operator with an oscillatory kernel will be called oscillatory ; G_1 and G_2 are oscillatory operators.

Further on we shall require the following Lemma.

L e m m a 2.1 . The operators G_1 and G_2 in the strip $|\mathrm{Im}\,\alpha| < \delta_0$ are analytic functions of the parameter α , i.e. in the neighborhood of any α from this strip they can be expanded in Taylor series which converge in the norm of operators. The positive number δ_0 depends solely on r_1 and r_2.

P r o o f . Let us make use of the familiar fact (e.g. see [8]) that if a linear operator depends analytically on a parameter within some range of its variation and has an inverse operator, then the inverse operator is an analytic function of the parameter in the same range.

The boundary value problems

$$(L-\alpha^2)v = -f, \quad v = 0 \quad \text{for } r = r_1, r_2$$
$$(L-\alpha^2)^2 u = f, \quad u = u' = 0 \quad \text{for } r = r_1', r_2 \tag{3.2}$$

are equivalent, respectively, to the integral equations

$$v + \alpha^2 G_{10} v = G_{10} f, \qquad u - 2\alpha^2 G_{20} L u + \alpha^4 G_{20} u = G_{20} f \tag{3.3}$$

where G_{k0} ($k = 1, 2$) means the operator G_k for $\alpha = 0$.

From (3.3) we have the following representations of the operators G_1 , G_2:

$$G_1 = (I + \alpha^2 G_{10})^{-1} G_{10}, \qquad G_2 = (I - 2\alpha^2 G_{20} L + \alpha^4 G_{20})^{-1} G_{20} \tag{3.4}$$

We note that the operators G_{10} , G_{20} are completely continuous, symmetrical and positive, and that the operator $G_{20}L$ admits of extension to a completely continuous operator (integration by parts transforms it into an integral operator with a continuous kernel).

It is now sufficient to establish that operators inverse to those of (3.4) exist for any α from some strip $|\mathrm{Im}\,\alpha| < \delta_0$. Let the eigenvalues of the operator G_{10} be $0 < \delta_1^2 < \delta_2^2 < \ldots < \delta_n^2 < \ldots$, and let $\delta_n^2 \to \infty$ as $n \to \infty$. Then the operator $(I + \alpha^2 G_{10})^{-1}$ exists for any α except $\alpha = \mp i\delta_1, \mp i\delta_2, \ldots$ and, in any case, for any α from the strip $|\mathrm{Im}\,\alpha| < \delta_1$. This proves the required statement about the operator G_1

We shall now show that if the quantity $|\mathrm{Im}\,\alpha|$ is sufficiently small, then the second equation of (3.3) for $f = 0$, or (equivalently), the second boundary value problem of (3.2) for $f = 0$ has a zero solution only. Multiplying the second equation of (3.2) for $f = 0$ by u^* and integrating over r from r_1 to r_2 we obtain

$$\| u \|_2^2 + 2\alpha^2 \| u \|_1^2 + \alpha^4 \| u \|_0^2 = 0 \tag{3.5}$$

$$\left(\| u \|_0 = \| u \|_{H_0}, \ \| u \|_2 = \| Lu \|_{H_0}, \ \| u \|_1^2 = \int_{r_1}^{r_2} \left| \frac{du}{dr} + \frac{u}{r} \right|^2 r dr \right)$$

Setting $\alpha = \gamma + i\delta$ and separating the real and imaginary parts in (3.5), we obtain

$$\| u \|_2^2 + 2(\gamma^2 - \delta^2) \| u \|_1^2 + [(\gamma^2 - \delta^2)^2 - 4\gamma^2\delta^2] \| u \|_0^2 = 0 \tag{3.6}$$
$$\gamma\delta \left[\| u \|_1^2 + (\gamma^2 - \delta^2) \| u \|_0^2 \right] = 0 \tag{3.7}$$

As we know (see [7]), the following inequalities are valid:

$$\| u \|_0 \leqslant c_0 \| u \|_1, \qquad \| u \|_1 \leqslant c_1 \| u \|_2 \tag{3.8}$$

where c_0, c_1 are positive constants which depend only on r_1 and r_2. We set

$$\delta_0 = \min\left\{\delta_1, \frac{1}{c_0}, \frac{1}{c_1\sqrt{2}}\right\}$$

and let $|\mathrm{Im}\,\alpha| = |\delta| < \delta_0$. Then the fact that $u = 0$ follows: (1) for $\delta = 0$ from (3.6),(2) for $\delta \neq 0$, $\gamma \neq 0$ from (3.7). In fact, from (3.7) and (3.8) we deduce that

$$0 = \|u\|_1^2 + (\gamma^2 - \delta^2)\|u\|_0^2 \geqslant (1 - \delta_0^2 c_0^2)\|u\|_1^2 + \gamma^2\|u\|_0^2 \geqslant \gamma^2\|u\|_0^2$$

3) for $\delta \neq 0$,, $\gamma = 0$ from (3.6). From (3.6),(3.8) for $\gamma = 0$ we find, in fact, that

$$0 = \|u\|_2^2 - 2\delta^2\|u\|_1^2 + \delta^4\|u\|_0^2 \geqslant (1 - 2\delta_0^2 c_1^2)\|u\|_2^2 + \delta^4\|u\|_0^2 \geqslant \delta^4\|u\|_0^2$$

The foregoing, in accordance with Fredholm's theorem, implies that for $|\mathrm{Im}\,\alpha| < \delta_0$ the operator $I - 2\alpha^2 G_{20}L + \alpha^4 G_{20}$ is invertible. The second expression of (3.4) implies that the operator G_2 in the strip $|\mathrm{Im}\,\alpha| < \delta_0$ depends analytically on α. The Lemma has been proved.

4. The spectrum of bifurcation. In the present section we shall establish conditions under which the operator A_0 defined in (1.11) has a real and singular spectrum. From this, by virtue of Krasnosel'skii's theorem, we deduce that each eigenvalue of the operator A_0 is a bifurcation point of the operator K_0: for values of the parameter λ which are close to it, Equation (1.10) and thereby boundary value problem (1.6),(1.7), have zero solutions.

T h e o r e m 4.1 . Let the conditions

$$\omega(r) = v_0(r)\,/\,r > 0 \qquad (r_1 < r < r_2) \tag{4.1}$$

$$g(r) = -(dv_0\,/\,dr + v_0\,/\,r) > 0 \qquad (r_1 < r < r_2) \tag{4.2}$$

be fulfilled

Then for any α_0 with the exception of some denumerable set the operator A_0 has a sequence of positive and simple eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ each of which is a bifurcation point of Equation (1.10). All of the intervals (λ_1, λ_2), (λ_3, λ_4) ... belong to the spectrum of Equation (1.10).

If condition (4.1) is fulfilled, and if the inequality

$$g(r) = -(dv_0\,/\,dr + v_0\,/\,r) < 0 \tag{4.3}$$

which is the opposite of (4.2) is satisfied instead of the latter, then the operator A_0 does not have any real eigenvalues, and there is no bifurcation.

P r o o f . Let conditions (4.1) and (4.2) be fulfilled. Since the product of the oscillatory operators is yet another oscillatory operator [3 to 5], the operator B defined in (2.11) is oscillatory. According to the results of [4 and 5] this implies that its spectrum consists of a sequence of singular positive eigenvalues $0 < \mu_1(\alpha) < \mu_2(\alpha) < \ldots < \mu_n(\alpha) < \cdots$ The spectrum of the operator A_0 therefore consists of the real eigenvalues

$$\lambda_{ik} = \left(\frac{\mu_i(k\alpha_0)}{2k^2\alpha_0^2}\right)^{1/2}, \qquad \lambda_{ik}' = -\left(\frac{\mu_i(k\alpha_0)}{2k^2\alpha_0^2}\right)^{1/2} \qquad (i, k = 1, 2, \ldots)$$

According to Lemma 1.1, all of them have a rank of 1 . For this reason the multiplicity of each of them is equal to the dimensionality of the free vector space. Thus, the multiplicity of an eigenvalue, let us say $\lambda_{i_0 k_0}$, is equal to the number of elements in the matrix (λ_{ik}) situated in the same row and equal to $\lambda_{i_0 k_0}$ (because of the singularity of eigenvalues μ_i, the columns of the matrix (λ_{ik}) do not contain identical elements).

What follows is based on the following Lemma, which is a simple corollary of the theory of perturbations of the spectrum of a linear operator.

L e m m a 4.1 . Let the linear operator $B(\alpha)$ be completely continuous and analytic with respect to α along the real axis. Let its spectrum consist of a sequence of positive and single eigenvalues $0 < \mu_1(\alpha) < \mu_2(\alpha) < \ldots$ $\ldots < \mu_n(\alpha) < \ldots$ Then all $\mu_k(\alpha)$ $(k = 1, 2, \ldots)$ are analytic functions on the real axis.

P r o o f . According to perturbation theory (e.g. see [6 to 8]), the singularity of the eigenvalue $\mu_k(\alpha)$ implies the possibility of its analytic extension along α . In this case $\mu_{k-1}(\alpha) < \mu_k(\alpha) < \mu_{k+1}(\alpha)$ $(k = 2, 3, \ldots)$, is valid for any α , since it cannot be violated without the appearance of a multiple eigenvalue for some value of α . The Lemma has been proved.

It is sufficient to consider the eigenvalues λ_{ik}.

Let us set $\Lambda_i(\alpha) = \sqrt{\mu_i(\alpha)/\alpha^2}$. (The function $\Lambda_i(\alpha)$ is analytic with respect to α on the positive semiaxis $\alpha > 0$. We have $\lambda_{ik} = \Lambda_i(k\alpha_0)$. The set Γ of those α_0 for which there are at least two identical numbers among the λ_{ik} is clearly the join over all natural i, k, r, s of the sets Γ_{ikrs} of those α_0 for which the equation

$$\Lambda_i(k\alpha_0) - \Lambda_r(s\alpha_0) = 0 \tag{4.4}$$

is fulfilled.

We shall show that the analytic function $\Lambda_{ikrs}(\alpha) = \Lambda_i(k\alpha) - \Lambda_r(s\alpha)$ cannot be an identical zero.

In fact, setting for example $i > r$ and taking account of the inequality $\Lambda_i(\alpha) > \Lambda_r(\alpha)$, from the assumption that $\Lambda_{ikrs} \equiv 0$ we arrive at the conclusion that

$$\Lambda_r(s\alpha) = \Lambda_i(k\alpha) > \Lambda_r(k\alpha)$$

which is impossible for $s = k$. If, on the other hand, $s \neq k$, then for any $0 < \alpha < \infty$ and any natural p we have

$$\Lambda_r(\alpha) > \Lambda_r\left(\left(\frac{k}{s}\right)^p \alpha\right) \to +\infty \qquad \text{for} \quad p \to \infty$$

since $\Lambda_r(\alpha) \to \infty$ as $\alpha \to 0$ or ∞ (see Note 2 below). The same reasoning applies to the case $i = r$.

Now we can say that Γ_{ikrs} as the set of zeros of the analytic function Λ_{ikrs} is not more than denumerable. Hence the set Γ is not more than denumerable.

Γ_{ikrs} is, in fact, denumerable; its limiting points are 0 and ∞ (see Note 3 below).

Let $\alpha_0 \bar{\in} \Gamma$. We number the eigenvalues λ_{ik} in increasing order to obtain the sequence of single eigenvalues of the operator A_0 : $0 < \lambda_1 < \lambda_2 < \ldots$. Each of them, in accordance with Krasnosel'skii's theorem [1], is a bifurcation point of Equation (1.10).

From the results of [9] it follows that the rotation of the vector field $(I - \lambda K_0)v$ on spheres of large radius in the space H_1° is equal to $+1$. For this reason, just as in [2] we find that equation (1.10) has nontrivial solutions for any λ from the intervals $(\lambda_1, \lambda_2), (\lambda_3, \lambda_4) \ldots$. These intervals belong to the spectrum of the operator K_0.

If conditions (4.1) and (4.3) are fulfilled, then the operator $-B$ is oscillatory. Hence, the operator A_0 has no real eigenvalues under these conditions (they are all imaginary), and there is no bifurcation. The theorem has been proved.

N o t e 1 . The theorem and its proof remain valid if equality in individual points of the segment (r_1, r_2) is admitted in (4.1) and (4.2). We can also consider very irregular ω and g, e.g. replacing $grdr$ by $d\sigma$ in the second integral operator of (2.10); here σ is an arbitrary increasing function.

N o t e 2 . Let us call the quantity

$$\Lambda_0 = \min \Lambda_1(\alpha) \qquad \text{for } 0 < \alpha < \infty$$

the bifurcational critical Reynolds number.

We shall prove that $\Lambda_0 > 0$ and is attained for some value of α.

Since the function $\Lambda_1(\alpha)$ is continuous for $\alpha \in (0, \infty)$, it is sufficient to establish that $\Lambda_1(\alpha) \to +\infty$ for $\alpha \to 0, \infty$. Since $\Lambda_1(\alpha) = \alpha^{-1}\sqrt{\mu_1(\alpha)}$, and $\mu_1(\alpha)$ is an analytic function on the entire real axis, and $\mu_1(0) > 0$, it follows that

$$\Lambda_1(\alpha) \sim \alpha^{-1}\sqrt{\mu_1(0)} \to +\infty \qquad \text{for} \quad \alpha \to 0$$

Further, multiplying the first two equations of (2.2) for $\lambda = \Lambda_1(\alpha)$ by ru, $-rv$, respectively, and integrating over $[r_1, r_2]$, we obtain the relations (for notation see (3.5))

$$\|u\|_2^2 + 2\alpha^2\|u\|_1^2 + \alpha^4\|u\|_0^2 = 2\alpha^2\Lambda_1 \int_{r_1}^{r_2} \omega vurdr, \quad \|v\|_1^2 + \alpha^2\|v\|_0^2 = \Lambda_1 \int_{r_1}^{r_2} guvrdr \qquad (4.5)$$

We set

$$C = \max \{\max_{r_1 \leqslant r \leqslant r_2} |\omega(r)|, \ \max_{r_1 \leqslant r \leqslant r_2} |g(r)|\}$$

Applying the Buniakowski inequality, we find from (4.5) that

$$\alpha^4 \|u\|_0^2 \leqslant 2\Lambda_1 C \|u\|_0 \|v\|_0, \qquad \alpha^2 \|v\|_0^2 \leqslant \Lambda_1 C \|u\|_0 \|v\|_0 \qquad (4.6)$$

By virtue of the fact that (u, v) is a nonzero solution, we obtain from (4.6) the estimate

$$\Lambda_1 \geqslant \sqrt{2}\alpha^2/C \to +\infty \qquad \text{for} \quad \alpha \to +\infty$$

This proves our statement.

It remains uncertain whether it is true that flow (1.2) is stable for $\lambda < \Lambda_0$ ("principle of alteration of stability").

N o t e 3 . The exclusive denumerable set Γ mentioned in the proof of the theorem actually does exist.

In fact, since the function $\lambda = \Lambda_1(\alpha) \to \infty$ as $\alpha \to 0, \infty$, we can point out two continuous branches of the inverse function $\alpha_1(\lambda)$, $\alpha_2(\lambda)$ such that $\alpha_1(\lambda) \to 0$; $\alpha_2(\lambda) \to \infty$ as $\lambda \to +\infty$. Then $\alpha_2(\lambda)/\alpha_1(\lambda) \to +\infty$ as $\lambda \to \infty$, so that for each natural k starting with some particular k there exists an η_k such that $\alpha_2(\eta_k) = k\alpha_1(\eta_k)$. Let us set $\alpha_1(\eta_k) = \alpha_{0k}$. Then $\Lambda_1(k\alpha_{0k}) = \Lambda_1(\alpha_{0k})$. the set Γ contains all of the points α_{0k} $(k = 1, 2, \ldots)$ and is therefore denumerable.

Approximate computations indicate (it would be interesting to have a rigorous proof of this) that $\Lambda_1(\alpha)$ is a downward convex function. If this is indeed the case, it follows that Λ_0 is attained for a unique value of α_0, and that $\lambda_1(\alpha_0)$ is a bifurcation point in a certain one of its neighborhoods.

N o t e 4 . The theory of oscillatory operators [3 and 5] implies several things about the properties of the eigensolutions of problems (2.2).

For example, for solutions (u, v), (u_1, v_1) of these problems which correspond to the minimum eigenvalue, all of the functions u, v, u_1, v_1 retain the same sign on the segment (r_1, r_2), while for the kth (in magnitude) eigenvalue, each of these functions changes sign $k-1$ times. The eigenvalues of each of the equations (2.6),(2.9) from a complete system in H_0.

N o t e 5. As an example, let us consider Couette flow (1.4). Here we have

$$\omega(r) = a + b / r^2, \quad g(r) = -2a \tag{4.7}$$

From Theorem 4.1 we find that if the cylinders are rotating in the same direction, $\omega_1 > 0$, $\omega_2 \geq 0$ (the outer cylinder can be at rest), then secondary steady flows arise with a certain Reynolds number, provided the condition

$$\omega_2 r_2^2 - \omega_1 r_1^2 < 0 \tag{4.8}$$

On the other hand, if the opposite inequality

$$\omega_2 r_2^2 - \omega_1 r_1^2 > 0 \tag{4.9}$$

applies, then there is no bifurcation. As we know (see [10 and 11]) in this case the Couette flow is stable relative to axisymmetrical perturbations for any Reynolds number.

Let us cite a simple proof of this fact (*).

We shall consider the nonsteadystate equations corresponding to system (1.8). These are obtained by adding the terms $-\lambda(\partial u_r / \partial t)$, $-\lambda(\partial u_\theta / \partial t)$, $-\lambda(\partial u_z / \partial t)$ to the left-hand sides of the second, third, and fourth equations of (1.8), respectively. Multiplying these equations by u_r, $-2\omega / g\,(u_\theta)$, u_z, summing the resulting equations, and integrating over the domain $D\,(r_1 < r < r_2;\ |z| < \pi / a_0)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_D (u_r^2 + h u_\theta^2 + u_z^2)\, r dr dz = -\nu \int_D \left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_r}{\partial z}\right)^2 + \frac{u_r^2}{r^2} + \left(\frac{\partial u_z}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] r dr dz - \nu \int_D \left[h\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2 + \left(\frac{dh}{dr} + \frac{2h}{r}\right) v_\theta \frac{\partial v_\theta}{\partial r} + h\left(\frac{\partial v_\theta}{\partial z}\right)^2\right] r dr dz \qquad \left(h = -\frac{2\omega}{g}\right) \tag{4.10}$$

In order for the right-hand side in (4.10) to be negative, and for flow (1.4) to be stable for any Reynolds number, it is sufficient that the functional

$$l(v) = \int_{r_1}^{r_2} \left[h\left(v' - \frac{v}{r}\right)^2 + \left(\frac{dh}{dr} + \frac{2h}{r}\right) v v'\right] r dr \tag{4.11}$$

be nonnegative on the set of smooth functions $v(r)$ which vanish for $r = r_1, r_2$. Specifically, this requires that the function h be nonnegative. But for Couette flow with allowance for (4.7) and (4.9) we find with $a > 0$ that

$$h = -\frac{2\omega}{g} = 1 + \frac{b}{ar^2} \geqslant 0, \qquad l(v) = \int_{r_1}^{r_2} h\left(v' - \frac{v}{r}\right)^2 r dr > 0 \qquad (v' \neq 0)$$

*) A slight alteration of this proof would enable us to show that stability also takes place when $\omega_2 r_2^2 - \omega_1 r_1^2 = 0$.

We have just proved the stability of flow (1.3) in a linear approximation. However, according to the results of [12], nonlinear stability also follows.

5. Instability. In this section we shall establish that under conditions (4.1) and (4.2) which ensure the appearance of secondary flows, principal flow (1.2) is unstable for sufficiently large Reynolds numbers.

In [13] this fact was established in the case of Couette flow through the asymptotic integration of system (5.1),(5.2) for $\lambda \to \infty$.

As we know, the matter is reduced to an investigation of the spectrum of the boundary value problem

$$(L-\alpha^2)^2 u - \sigma (L-\alpha^2) u = 2\alpha^2\lambda\omega v, \qquad (L-\alpha^2) v - \sigma v = -\lambda g u \quad (5.1)$$

$$u = u' = 0 \ (r = r_1, r_2), \qquad v = 0 \ (r = r_1, r_2) \quad (5.2)$$

If all of the eigenvalues σ_k ($k = 1, 2, \ldots$) for a given λ have negative real parts, then flow (1.2) is stable. The existence of at least one eigenvalue with a positive real part results in instability. The applicability of the method to the nonlinear instability problem is justified in [12].

T h e o r e m 5.1 . Let conditions (4.1) and (4.2) be fulfilled. Then for any $\sigma > -(\alpha^2 + \sigma_0)$ ($\sigma_0 > 0$ depends only on r_1 and r_2) there exists a sequence $\lambda_1 < \lambda_2 < \ldots$; $\lambda_n \to \infty$ of λ values such that problem (5.1),(5.2) has a nontrivial solution.

P r o o f . The differential operators in the left-hand sides of Equations (5.1) admit of the representation

$$(L-\alpha^2)^2 u - \sigma (L-\alpha^2) u = \frac{1}{r} [\rho_0 \frac{d}{dr} \rho_1 \frac{d}{dr} \rho_2 \rho_{0\sigma} \frac{d}{dr} \rho_{1\sigma} \frac{d}{dr} \rho_{2\sigma}] u$$

$$(L-\alpha^2) v - \sigma v = \rho_{0\sigma} \frac{d}{dr} \rho_{1\sigma} \frac{d}{dr} \rho_{2\sigma} v \quad (5.3)$$

where ρ_0, ρ_1, ρ_2 are functions defined in (3.1), and $\rho_{0\sigma}$, $\rho_{1\sigma}$, $\rho_{2\sigma}$ are given by Equations

$$r\rho_{0\sigma} = \rho_{2\sigma} = Y_1(r), \qquad \rho_{1\sigma} = \frac{r}{\rho_{2\sigma}^2} \quad (5.4)$$

where $Y_1(r)$ is some solution of Equation

$$(L - \alpha^2 - \sigma) Y_1 = 0 \quad (5.5)$$

If $\alpha^2 + \sigma > -\sigma_0$, where σ_0 is the first eigenvalue of the differential operator $-L$ for the second condition of (5.2), then Equation (5.5) has a solution Y_1 which is positive on the segment $[r_1, r_2]$ (if $\alpha^2 + \sigma = \beta^2 \geq 0$, as our Y_1 we simply take $I_1(\beta_r)$).

By virtue of the results of Krein [3 and 4], (5.3) implies that the corresponding Green's operators $G_{1\sigma}$, $G_{2\sigma}$ are oscillatory. It remains for us to note that boundary value problem (5.1),(5.2) is equivalent to the integral equation

$$u = \mu B_\sigma u, \quad B_\sigma = G_{2\sigma} \omega G_{1\sigma} g \quad (5.6)$$

with the oscillatory operator B_σ and once again refer the reader to the results of [5]. The theorem has been proved.

BIBLIOGRAPHY

1. Krasnosel'skii, M.A., Topologicheskie metody v teorii nelineynykh integral'nykh uravnenii (Topological Methods in the Theory of Nonlinear Integral Equations). Gostekhizdat, Moscow, 1956.

2. Iudovich, V.I., Primer rozhdeniia vtorichnogo statsionarnogo ili periodicheskogo techeniia pri potere ustoichivosti laminarnogo techeniia viazkoi neszhimaemoi zhidkosti (Example of the generation of the secondary steady or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid). *PMM* Vol.29, № 3, 1965.

3. Gantmakher, F.R. and Krein, M.G., Ostsilliatsionnye matritsy i iadra i malye kolebaniia mekhanicheskikh sistem (Oscillatory matrices and kernels and small oscillations of mechanical systems). GITTL, Moscow-Leningrad, 1950.

4. Krein, M.G., O nesimmetricheskikh ostsilliatsionnykh funktsiiakh Grina obyknovennykh differentsial'nykh operatorov (On asymmetrical oscillatory Green's functions of ordinary differential operators). Dokl. Akad.Nauk SSSR, Vol.25, № 8, 1939.

5. Gantmakher, F.R., O nesimmetrichnykh iadrakh Kelloga (On asymmetrical Kellog kernels). Dokl.Akad.Nauk SSSR, Vol. 1(X), № 1, 1936.

6. Vainberg, M.M. and Trenogin, V.A., Metody Liapunova i Shmidta v teorii nelineinykh uravnenii i ikh dal'neishee razvitie (The Liapunov and Schmidt methods in the theory of nonlinear equations and their further development). Usp.mat.Nauk, Vol.17, № 2 (104), 1962.

7. Mikhlin, S.G., Variatsionnye metody v matematicheskoi fizike (Variational methods in mathematical physics). GITTL, Moscow, 1957.

8. Danford, N. and Schwartz, J.T., Lineinye operatory (Linear Operators). Izd.inostrannoi literatury, Moscow, 1962.

9. Vorovich, I.I. and Iudovich, V.I., Statsionarnoe techenie viazkoi neszhimaemoi zhidkosti (Steady flow of a viscous incompressible fluid). Mat.Sb., Vol.53 (95), № 4, 1961.

10. Lin' Tzia-tsziao (Lin Chia-chiao), Teoriia gidrodinamicheskoi ustoichivosti (Theory of Hydrodynamic Stability). Izd.inostrannoi literatury, Moscow, 1958.

11. Synge, J.L., On the stability of a viscous liquid between two rotating coaxial cylinders. Proc.R.Soc., (London), Ser.A, Vol.167, 1938.

12. Iudovich, V.I., Ob ustoichivosti statsionarnykh techenii viazkoi neszhimaemoi zhidkosti (On the stability of the steady flows of a viscous incompressible fluid). Dokl.Akad.Nauk SSSR, Vol.161, № 5, 1965.

13. Krylov, A.L., Dokazatel'stvo neustoichivosti odnogo techeniia viazkoi neszhimaemoi zhidkosti (Proof of the instability of one of the flows of a viscous incompressible fluid). Dokl.Akad.Nauk SSSR, Vol.153, № 4, 1963.

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